

# BALANCING UNIT VECTORS

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ABSTRACT.

**Theorem A.** Let  $x_1, \dots, x_{2k+1}$  be unit vectors in a normed plane. Then there exist signs  $\varepsilon_1, \dots, \varepsilon_{2k+1} \in \{\pm 1\}$  such that  $\|\sum_{i=1}^{2k+1} \varepsilon_i x_i\| \leq 1$ .

We use the method of proof of the above theorem to show the following point facility location result, generalizing Proposition 6.4 of Y. S. Kupitz and H. Martini (1997).

**Theorem B.** Let  $p_0, p_1, \dots, p_n$  be distinct points in a normed plane such that for any  $1 \leq i < j \leq n$  the closed angle  $\angle p_i p_0 p_j$  contains a ray opposite some  $\overrightarrow{p_0 p_k}$ ,  $1 \leq k \leq n$ . Then  $p_0$  is a Fermat-Torricelli point of  $\{p_0, p_1, \dots, p_n\}$ , i.e.  $x = p_0$  minimizes  $\sum_{i=0}^n \|x - p_i\|$ .

We also prove the following dynamic version of Theorem A.

**Theorem C.** Let  $x_1, x_2, \dots$  be a sequence of unit vectors in a normed plane. Then there exist signs  $\varepsilon_1, \varepsilon_2, \dots \in \{\pm 1\}$  such that  $\|\sum_{i=1}^{2k} \varepsilon_i x_i\| \leq 2$  for all  $k \in \mathbf{N}$ .

Finally we discuss a variation of a two-player balancing game of J. Spencer (1977) related to Theorem C.

## 1. INTRODUCTION

In this note we consider balancing results for unit vectors related to work of Bárány and Grinberg [1], Spencer [6] and Peng and Yan [5]. We apply these results to generalize a point facility location result from the Euclidean plane [4] to general normed planes. Finally we consider a dynamical balancing problem for unit vectors in the form of a two-player perfect information game. Our results will mainly be in a normed plane  $X$  with norm  $\|\cdot\|$  (except in Theorem 5, where higher-dimensional normed spaces are also considered).

**1.1. Balancing Unit Vectors.** Bárány and Grinberg [1] proved the following:

**Theorem 1** ([1]). *Let  $x_1, x_2, \dots, x_n$  be a sequence of vectors of norm  $\leq 1$  in a  $d$ -dimensional normed space. Then there exist signs  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{\pm 1\}$  such that*

$$\left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \leq d.$$

We sharpen this theorem for an odd number of *unit* vectors in a normed plane as follows.

**Theorem 2.** *Let  $x_1, \dots, x_{2k+1}$  be unit vectors in a normed plane. Then there exist signs  $\varepsilon_1, \dots, \varepsilon_{2k+1} \in \{\pm 1\}$  such that*

$$\left\| \sum_{i=1}^{2k+1} \varepsilon_i x_i \right\| \leq 1.$$

This result is best possible in any norm, as is seen by letting  $x_1 = x_2 = \dots = x_{2k+1}$  be any unit vector. The proof of this theorem is in Section 2. The method of proof can also be used to generalize a result on Fermat-Toricelli points from the Euclidean plane to an arbitrary normed plane (Section 1.3).

Bárány and Grinberg also proved the following dynamic balancing theorem.

**Theorem 3** ([1]). *Let  $x_1, x_2, \dots$  be a sequence of vectors of norm  $\leq 1$  in a normed space. Then there exist signs  $\varepsilon_1, \varepsilon_2, \dots \in \{\pm 1\}$  such that for all  $k \in \mathbf{N}$ ,*

$$\left\| \sum_{i=1}^k \varepsilon_i x_i \right\| \leq 2d.$$

Again, for unit vectors in a normed plane we sharpen this result as follows.

**Theorem 4.** *Let  $x_1, x_2, \dots$  be a sequence of unit vectors in a normed plane. Then there exist signs  $\varepsilon_1, \varepsilon_2, \dots \in \{\pm 1\}$  such that for all  $k \in \mathbf{N}$ ,*

$$\left\| \sum_{i=1}^{2k} \varepsilon_i x_i \right\| \leq 2.$$

*In the Euclidean plane the upper bound 2 can be replaced by  $\sqrt{2}$ .*

This result is best possible in the rectilinear plane with unit ball a parallelogram — let  $x_{2i-1} = e_1$  and  $x_{2i} = e_2$  for all  $i \in \mathbf{N}$ , where  $e_1$  and  $e_2$  are any adjacent vertices of the unit ball. See Section 3 for a proof of this theorem.

**1.2. Balancing Games.** Theorem 4 can be used to analyze the following variation of a two-player balancing game of Spencer. Fix  $k \in \mathbf{N}$  and a normed space  $X$ . Let the starting position of the game be  $p_0 = o \in X$ . In round  $i$ , Player I chooses  $k$  unit vectors  $x_1, \dots, x_k$  in  $X$ , and then Player II chooses signs  $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$ . Then the position is adjusted to  $p_i := p_{i-1} + \sum_{j=1}^k \varepsilon_j x_j$ .

**Theorem 5.** *In the above game, Player II can keep the sequence  $(p_i)_{i \in \mathbf{N}}$  bounded iff  $X$  is at most two-dimensional and  $k$  is even. In fact, Player II can force  $\|p_i\| \leq 2$  for all  $i \in \mathbf{N}$ .*

The proof is in Section 3. In [5] a vector balancing game with a buffer is considered. Theorem 5 readily implies Theorem 4 of [5] in the special case of unit vectors in a normed plane.

**1.3. Fermat-Toricelli points.** A point  $p$  in a normed space  $X$  is a *Fermat-Toricelli point* of  $x_1, x_2, \dots, x_n \in X$  if  $x = p$  minimizes  $x \mapsto \sum_{i=1}^n \|x_i - x\|$ . See [4] for a survey on the problem of finding such points. It is well-known that in the Euclidean plane, if  $x_1$  is in the convex hull of non-collinear  $\{x_2, x_3, x_4\}$ , then  $x_1$  is the (unique) Fermat-Toricelli point of  $x_1, x_2, x_3, x_4$ . Cieslik [2] generalized this result to an arbitrary normed plane (where the Fermat-Toricelli point is not necessarily unique). There is also a generalization by Kupitz and Martini [4, Proposition 6.4] in another direction.

**Theorem.** *Let  $p_0, p_1, \dots, p_{2m+1}$  be distinct points in the Euclidean plane such that for any distinct  $i$  and  $j$  the open angle  $\angle p_i p_0 p_j$  contains a ray opposite some  $\overrightarrow{p_0 p_k}$ ,  $1 \leq k \leq 2m+1$ . Then  $p_0$  is the unique Fermat-Toricelli point of  $\{p_0, p_1, \dots, p_n\}$ .*

We generalize this result as follows to an arbitrary normed plane.

**Theorem 6.** *Let  $p_0, p_1, \dots, p_n$  be distinct points in a normed plane such that for any distinct  $i$  and  $j$  the closed angle  $\angle p_i p_0 p_j$  contains a ray opposite some  $\overrightarrow{p_0 p_k}$ ,  $1 \leq k \leq n$ . Then  $p_0$  is a Fermat-Toricelli point of  $\{p_0, p_1, \dots, p_n\}$ .*

The proof is in Section 2. Our seemingly weaker hypotheses easily imply that  $n$  must be odd. The proof in [4] of the Euclidean case uses rotations. Our proof for any norm shows that it is really an affine result. The correct affine tool turns out to be the fact that two-dimensional centrally symmetric polytopes are zonotopes.

## 2. ZONOGONS

A *zonotope*  $P$  in a  $d$ -dimensional vector space  $X$  is a Minkowski sum of line segments

$$P = [x_1, y_1] + [x_2, y_2] + \dots + [x_n, y_n]$$

where  $x_1, \dots, x_n, y_1, \dots, y_n \in X$ . It is well-known that any centrally symmetric two-dimensional polytope (or polygon) is always a zonotope (or *zonogon*) [8, Example 7.14]. In particular, if  $x_1, \dots, x_n$  are consecutive edges of a  $2n$ -gon  $P$  symmetric around 0, then

$$(1) \quad P = \sum_{i=1}^n [(x_{i+1} - x_i)/2, (x_i - x_{i+1})/2]$$

where we take  $x_{n+1} = -x_1$ .

**Lemma 7.** *Let  $n \in \mathbb{N}$  be odd and let  $P$  be a polygon with vertices  $\pm x_1, \dots, \pm x_n$  with  $x_1, \dots, x_n$  in this order on the boundary of  $P$ . Then*

$$\sum_{i=1}^n (-1)^i x_i = \frac{1}{2} \sum_{i=1}^n (-1)^{i+1} (x_{i+1} - x_i) \in P.$$

*Proof.* The equation is simple to verify. That the right-hand side is in  $P$  follows from (1).  $\square$

Note that Lemma 7 does not hold for even  $n$ . We can now easily prove Theorem 2.

*Proof of Theorem 2.* Fix a line through the origin not containing any  $x_i$ . Fix one of the open half planes  $H$  bounded by this line. Then for each  $i$ ,  $\delta_i x_i \in H$  for some  $\delta_i \in \{\pm 1\}$ . We may renumber  $x_1, \dots, x_n$  such that  $\delta_1 x_1, \dots, \delta_n x_n$  occur in this order on  $P = \text{conv} \{\pm x_i\}$ . Now take  $\varepsilon_i = (-1)^i \delta_i$  and apply Lemma 7, noting that  $P$  is contained in the unit ball.  $\square$

Recall that the dual of a finite dimensional normed space  $X$  is the normed space of all linear functionals on  $X$  with norm  $\|\phi\| = \max\{\phi(u) : \|u\| = 1\}$ . A *norming functional*  $\phi$  of a non-zero  $x \in X$  is a linear functional satisfying  $\|\phi\| = 1$  and  $\phi(x) = \|x\|$ . Recall that by the separation theorem any non-zero  $x \in X$  has a norming functional (see e.g. [7]).

The following lemma is well-known and easily proved. See [4] for the Euclidean case and [3] for the general case. We only need the second case of the lemma, but we also state the first case for the sake of completeness.

**Lemma 8.** *Let  $p_0, p_1, \dots, p_n$  be distinct points in a finite-dimensional normed space  $X$ .*

- (1) *Then  $p_0$  is a Fermat-Toricelli point of  $p_1, \dots, p_n$  iff  $p_i - p_0$  has a norming functional  $\phi_i$  ( $1 \leq i \leq n$ ) such that  $\sum_{i=1}^n \phi_i = 0$ ,*
- (2) *and  $p_0$  is a Fermat-Toricelli point of  $p_0, p_1, \dots, p_n$  iff  $p_i - p_0$  has a norming functional  $\phi_i$  ( $1 \leq i \leq n$ ) such that  $\|\sum_{i=1}^n \phi_i\| \leq 1$ .*

*Proof of Theorem 6.* By Lemma 8 it is sufficient to find norming functionals  $\phi_i$  of  $p_i - p_0$  such that  $\|\sum_{i=1}^n \phi_i\| \leq 1$ . We order  $p_1, \dots, p_n$  such that  $\overrightarrow{p_0 p_1}, \dots, \overrightarrow{p_0 p_n}$  are ordered counter-clockwise. If  $p_0 \in [p_i, p_j]$  for some  $1 \leq i < j \leq n$ , we may choose  $\phi_i = -\phi_j$ . We may therefore assume that  $p_0 \notin [p_i, p_j]$  for all distinct  $i, j$ . Thus for any  $i$ , the *open angle*  $\angle p_i p_0 p_{i+1}$  contains a ray opposite some  $\overrightarrow{p_0 p_k}$ . We now show that necessarily  $n$  is odd and  $k \equiv i + (n+1)/2 \pmod{n}$ . Since each open angle contains at least one  $-p_k$ , each open angle contains exactly one such  $-p_k$ , say  $-p_{k(i)}$ . The line through  $p_0$  and  $p_{k(i)}$  cuts  $\{p_1, \dots, p_n\}$  in two open half planes: One half plane contains as many open angles as points  $p_i$ . Thus  $n$  is odd, and  $k(i) \equiv i + (n+1)/2 \pmod{n}$ .

It is now possible to choose norming functionals  $\phi_i$  of each  $p_i - p_0$  such that  $\phi_1, -\phi_{m+1}, \phi_2, -\phi_{m+2}, \dots$  are consecutive vectors on the unit circle in the dual normed plane. It is therefore sufficient to prove that in any normed plane, if we choose unit vectors  $x_1, \dots, x_n$  such that  $x_1, \dots, x_n, -x_1, \dots, -x_n$  are in this order on the unit circle, then  $\|\sum_{k=1}^n (-1)^k x_k\| \leq 1$ . This follows at once from Lemma 7.  $\square$

### 3. ONLINE BALANCING

*Proof of Theorem 5.*  $\Rightarrow$  We assume that some inner product structure has been fixed on  $X$ .

If  $k$  is odd then in round  $i$  Player I chooses the  $k$  unit vectors all to be the same unit vector, orthogonal to  $p_{i-1}$ . Then, independent of the choice of signs by Player II, the Euclidean norm of  $p_i$  grows  $> c\sqrt{i}$ .

If  $k$  is even and  $X$  is at least three-dimensional, Player I finds unit vectors  $e_1$  and  $e_2$  such that  $e_1, e_2, p_{i-1}$  are mutually orthogonal, then in round  $i$  takes  $e_1$  for the first  $k-1$  unit vectors, and  $e_2$  for the last unit vector. Again the Euclidean norm of  $p_i$  will grow  $> c\sqrt{i}$ .

$\Leftarrow$  follows immediately from Lemmas 9 and 10 below.  $\square$

*Proof of Theorem 4.* follows immediately from the following two lemmas.  $\square$

**Lemma 9.** *Let  $w, a, b$  be vectors in a normed plane such that  $\|w\| \leq 2$ ,  $\|a\| = \|b\| = 1$ . Then there exist signs  $\delta, \varepsilon \in \{\pm 1\}$  such that  $\|w + \delta a + \varepsilon b\| \leq 2$ .*

*Proof.* If  $a = \pm b$ , then the lemma is trivial. So assume that  $a$  and  $b$  are linearly independent. Let  $w = \lambda a + \mu b$ . Without loss of generality we assume that  $\lambda, \mu \geq 0$ , and show that  $\|w - a - b\| \leq 2$ .

If  $\lambda = 0$ , then  $0 \leq \mu \leq 2$  and  $\|(\lambda - 1)a + (\mu - 1)b\| \leq \|a\| + \|(\mu - 1)b\| \leq 2$ . So we may assume that  $\lambda > 0$ , and similarly,  $\mu > 0$ . Then we can write  $a = -(\mu/\lambda)b + (1/\lambda)w$ . Taking norms we obtain  $1 = \|a\| \leq \mu/\lambda + 2/\lambda$ , and therefore,  $\lambda - \mu \leq 2$ . Similarly,  $\mu - \lambda \leq 2$ . So we already have  $|(\lambda - 1) - (\mu - 1)| \leq 2$ . If furthermore  $\lambda + \mu \leq 4$ , we also obtain  $|(\lambda - 1) + (\mu - 1)| \leq 2$ , giving  $\|(\lambda - 1)a + (\mu - 1)b\| \leq |\lambda - 1| + |\mu - 1| \leq 2$ .

In the remaining case  $\lambda + \mu \geq 4$  we write  $(\lambda - 1)a + (\mu - 1)b$  as a non-negative linear combination

$$(\lambda - 1)a + (\mu - 1)b = \frac{\lambda + \mu - 4}{\lambda + \mu - 2}(\lambda a + \mu b) + \frac{2 + \lambda - \mu}{\lambda + \mu - 2}a + \frac{2 - \lambda + \mu}{\lambda + \mu - 2}b,$$

and apply the triangle inequality:

$$\|(\lambda - 1)a + (\mu - 1)b\| \leq 2 \frac{\lambda + \mu - 4}{\lambda + \mu - 2} + \frac{2 + \lambda - \mu}{\lambda + \mu - 2} + \frac{2 - \lambda + \mu}{\lambda + \mu - 2} = 2.$$

$\square$

**Lemma 10.** *Let  $w, a, b$  be vectors in the Euclidean plane such that  $\|w\| \leq \sqrt{2}$ ,  $\|a\| = \|b\| = 1$ . Then there exist signs  $\delta, \varepsilon \in \{\pm 1\}$  such that  $\|w + \delta a + \varepsilon b\| \leq \sqrt{2}$ .*

*Proof.* Note that  $a + b \perp a - b$ . Write  $p = a + b$ ,  $q = a - b$ . Let  $m$  be the midpoint of  $pq$ , and  $L$  the perpendicular bisector of  $pq$ . Assume without loss that  $\|p\| \geq \|q\|$  and that  $w$  is inside  $\angle poq$ . We now show that  $\|w - p\| \leq \sqrt{2}$  or  $\|w - q\| \leq \sqrt{2}$ . Note that as  $w$  varies,  $\min(\|w - p\|, \|w - q\|)$  is maximized on  $L$ . Let  $L$  and  $op$  intersect in  $c$  (between  $o$  and  $p$ ), and  $L$  and the circle with centre  $o$  and radius  $\sqrt{2}$  in  $d$  (inside  $\angle poq$ ). See Figure 1. Then clearly

$$\max_{\|w\| \leq \sqrt{2}} \min(\|w - p\|, \|w - q\|) = \max(\|p - c\|, \|p - d\|),$$

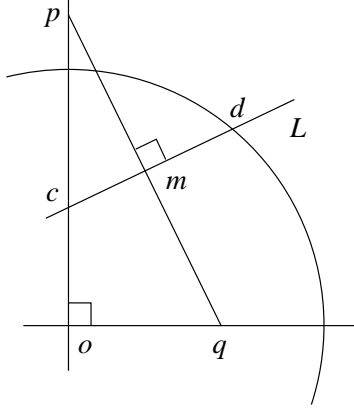


FIGURE 1.

and we have to show  $\|p - c\| \leq \sqrt{2}$  and  $\|p - d\| \leq \sqrt{2}$ . Since  $\|p\| \geq \|q\|$ , we have  $\angle opq \leq 45^\circ$  and  $\|p - c\| = \sec \angle opq \leq \sqrt{2}$ . Since  $c$  is between  $o$  and  $p$ , we have  $\angle omd \geq 90^\circ$ , hence  $\|m - d\|^2 \leq \|d\|^2 - \|m\|^2 = 2 - 1$ , and  $\|p - d\|^2 = \|p - m\|^2 + \|m - d\|^2 \leq 1 + 1$ .  $\square$

#### 4. CONCLUDING REMARKS

It would be interesting to find higher dimensional generalizations of our results and methods. We only make the following remarks.

Perhaps there is an analogue of Theorem 2 with an upper bound of  $d-1$  for  $n$  unit vectors in a  $d$ -dimensional normed space where  $n \not\equiv d \pmod{2}$ . This would be best possible, as the standard unit vectors in the  $d$ -dimensional space with the  $L_1$  norm show.

Regarding Theorem 4, it is not even clear what the best upper bound in Theorem 3 should be. Bárány and Grinberg [1] claim that they can replace  $2d$  by  $2d - 1$ . On the other hand, the upper bound cannot be smaller than  $d$ , as the  $d$ -dimensional  $L_1$  space shows [1]. As the negative part of Theorem 5 and the results of [5] show, an online method would have to have a (sufficiently large) buffer where Player II can put vectors supplied by Player I and take them out in any order.

We finally remark that a naive generalization of Theorem 6 is not possible, even in Euclidean 3-space. For example, using Lemma 8 it can be shown that for a regular simplex with vertices  $x_i$  ( $i = 1, \dots, 4$ ) there exists a point  $x_5$  in the interior of the simplex such that  $x_5$  is not a Fermat-Toricelli point of  $\{x_1, \dots, x_5\}$  — we may take any  $x_5$  sufficiently near a vertex.

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